

6.1. Nájďte intervaly konvexnosti a konkávnosti funkcie f . Určte inflexné body funkcie, ak existujú.

a) $f : y = x^3 - 9x^2 + 1$

h) $f : y = x + \frac{1}{x}$

b) $f : y = x^4 - 2x^3 - 7$

i) $f : y = 3x + \frac{1}{2x^2}$

c) $f : y = x^4 + 4x^3 - 18x^2 + 3x + 2$

j) $f : y = \frac{3x^2}{1-x}$

d) $f : y = x^4 - x^5$

k) $f : y = \frac{x^2 + x + 21}{x + 2}$

e) $f : y = \frac{x^6}{6} - \frac{x^5}{4} + 3$

l) $f : y = \frac{2x}{1+x^2}$

f) $f : y = 3x - (4-x)^5$

m) $f : y = \frac{x}{1-x^2}$

g) $f : y = x^4 + 2x^3 + 6x^2$

n) $f : y = \frac{x^2}{16-x^2}$

6.2. Nájďte intervaly konvexnosti a konkávnosti funkcie f . Určte inflexné body funkcie, ak existujú.

a) $f : y = \frac{x^2 + 1}{x^2 - 1}$

e) $f : y = 3x - \sqrt{x-3}$

b) $f : y = \frac{1}{x^3} + \frac{1}{x^2}$

f) $f : y = 4 + \sqrt[3]{x^2}$

c) $f : y = \frac{1}{x^3} - \frac{6}{x}$

g) $f : y = \frac{2x}{\sqrt{x^2 + 1}}$

d) $f : y = \left(\frac{1}{2} + \frac{1}{x}\right)^2$

h) $f : y = \frac{x}{\sqrt{x^3 + 1}}$

6.3. Nájďte intervaly konvexnosti a konkávnosti funkcie f . Určte inflexné body funkcie, ak existujú.

a) $f : y = x \cdot e^{-x}$

f) $f : y = e^{4 - \frac{x^2}{2}}$

b) $f : y = e^{-x^2}$

g) $f : y = e^{1 - \frac{x^3}{3}}$

c) $f : y = x \cdot e^{-x^2}$

h) $f : y = e^{2x} - 8e^x + 5x$

d) $f : y = x^2 \cdot e^{-x}$

i) $f : y = (2 - x^2) \cdot e^{-x}$

e) $f : y = x \cdot e^{\frac{1}{x}}$

j) $f : y = \frac{e^x}{x}$

Teória: Aplikácie derivácie funkcie:

1. Monotónnosť: pre všetky $x \in D(f)$ také, že $f'(x) > 0$ platí, že funkcia rastie
pre $\forall x \in (a, b): f'(x) < 0 \Rightarrow f(x)$ na (a, b) klesá

2. Konvexnosť, konkávnosť: pre $\forall x \in (a, b): f''(x) > 0 \Rightarrow f(x)$ je na (a, b) konvexná
pre $\forall x \in (a, b): f''(x) < 0 \Rightarrow f(x)$ je na (a, b) konkávna

3. Lokálne extrémum: pre $\forall x_0 \in D(f): f'(x_0) = 0 \Rightarrow f(x)$ má v bode x_0 lokálne extrémum
 x_0 sa nazýva stacionárny bod

ak $f''(x_0) > 0 \Rightarrow f(x)$ má v x_0 lokálne minimum

ak $f''(x_0) < 0 \Rightarrow f(x)$ má v x_0 lokálne maximum

4. Inflexné body: pre $\forall x_0 \in D(f): f''(x_0) = 0 \Rightarrow f(x)$ ~~sa~~ sa mení v x_0 z konvexnej na konkávnu
(alebo opačne)
 x_0 sa nazýva inflexný bod

5.3 m) $f: y = \frac{2}{x} + \ln x^2$; $D(f) = (-\infty; 0) \cup (0; \infty)$

$$y' = -2x^{-2} + \frac{1}{x^2} \cdot 2x = \frac{-2}{x^2} + \frac{2}{x} = \frac{-2+2x}{x^2} = \frac{2x-2}{x^2} > 0 \Leftrightarrow 2x-2 > 0 \quad | :2$$

$x-1 > 0$
 $x > 1 \Rightarrow$ na $(1; \infty)$ rastie
 na $(-\infty; 0)$ a na $(0; 1)$ klesá

m) $f: y = \frac{x}{\ln x}$; $D(f) = \underbrace{x > 0} \wedge \underbrace{\ln x \neq 0}_{x \neq 1} \Rightarrow D(f) = (0; 1) \cup (1; \infty)$

$$y' = \frac{1 \cdot \ln x - x \cdot \frac{1}{x}}{\ln^2 x} = \frac{\ln x - 1}{\ln^2 x} > 0 \Leftrightarrow \ln x - 1 > 0$$

$\ln x > 1$
 $x > e \Rightarrow$ na $(e; \infty)$ rastie
 na $(0; 1)$ a na $(1; e)$ klesá

6.1 a) $f: y = x^3 - 9x^2 + 1$

$$y' = 3x^2 - 18x$$

$$y'' = 6x - 18 = 0$$

$$6x = 18$$

$x = 3 \rightarrow$ inflexný bod

$$y'' > 0$$

$$6x - 18 > 0$$

$x > 3 \Rightarrow$ funkcia konvexná, inak konkávna

\Rightarrow na $(3; \infty)$ konvexná
 na $(-\infty; 3)$ konkávna
 $x_0 = 3$ je inflexný bod

6.1 b) $f: y = x^4 - 2x^3 - 7$

$y' = 4x^3 - 6x^2$

$y'' = 12x^2 - 12x = 0 \quad | :12$

$x(x-1) = 0 \rightarrow \underline{x_0 = 0 \wedge x_0 = 1}$

$y'' > 0$

$12x^2 - 12x > 0$

$x(x-1) > 0 \Leftrightarrow (x > 0 \wedge x > 1) \vee (x < 0 \wedge x < 1)$

$(1; \infty) \cup (-\infty; 0)$

$\Rightarrow x_0 = 0$ a $x_0 = 1$ sú inflexné body funkcie
 funkcia je na $(-\infty; 0)$ a na $(1; \infty)$ konvexná
 funkcia je na $(0; 1)$ konkávna

c) $f: y = x^4 + 4x^3 - 18x^2 + 3x + 2$

$y' = 4x^3 + 12x^2 - 36x + 3$

$y'' = 12x^2 + 24x - 36 = 0 \quad | :12$

$x^2 + 2x - 3 = 0$

$D = 4 + 4 \cdot 3 = 16$

$x_{1,2} = \frac{-2 \pm 4}{2} = \begin{matrix} -3 \\ 1 \end{matrix} \Rightarrow \underline{x_0 = -3 \wedge x_0 = 1}$

$y'' > 0$

$x^2 + 2x - 3 > 0$

$(x+3)(x-1) > 0$

$(x > -3 \wedge x > 1) \vee (x < -3 \wedge x < 1)$

$(1; \infty) \cup (-\infty; -3)$

\Rightarrow inflexné body: $x_0 = -3$ a $x_0 = 1$
 konvexná na $(-\infty; -3)$ a na $(1; \infty)$
 konkávna na $(-3; 1)$

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d) $f: y = x^4 - x^5$

$$y' = 4x^3 - 5x^4$$

$$y'' = 12x^2 - 20x^3 = 0 \quad | :4$$

$$x^2(3 - 5x) = 0 \Leftrightarrow x^2 = 0 \vee 3 - 5x = 0$$

$$\boxed{x_0 = 0 \vee x_0 = \frac{3}{5}}$$

e) $f: y = \frac{x^6}{6} - \frac{x^5}{4} + 3$

$$y' = \frac{1}{6} \cdot 6x^5 - \frac{1}{4} \cdot 5x^4 = x^5 - \frac{5}{4}x^4$$

$$y'' = 5x^4 - \frac{5}{4} \cdot 4x^3 = 5x^4 - 5x^3 = 5x^3(x-1) = 0 \Leftrightarrow \boxed{x_0 = 0 \vee x_0 = 1}$$

f) $f: y = 3x - (4-x)^5$

$$y'' = (y')' = [3 - 5 \cdot (4-x)^4 \cdot (-1)]' = (3 + 5 \cdot (4-x)^4)' = 20 \cdot (4-x)^3 \cdot (-1) = -20 \cdot (4-x)^3 = 0 \Leftrightarrow \boxed{x_0 = 4}$$

$$y'' > 0 \Leftrightarrow -20(4-x)^3 > 0 \Leftrightarrow \underbrace{-20 \cdot (4-x)^2}_{>0} \cdot (4-x) > 0 \Leftrightarrow -(4-x) > 0 \Leftrightarrow x-4 > 0 \Leftrightarrow \boxed{x > 4}$$

$$\Rightarrow \text{ib: } x_0 = 4$$

konvexná na $(4; \infty)$

konkávna na $(-\infty; 4)$

$$x^2 \cdot (3-5x) > 0 \Leftrightarrow 3-5x > 0 \Leftrightarrow x < \frac{3}{5} \quad [x^2 > 0 \text{ vždy}]$$

$$\Rightarrow \text{inflexné body: } x_0 = 0 \wedge x_0 = \frac{3}{5}$$

konvexná na $(-\infty; 0)$ a na $(0; \frac{3}{5})$

konkávna na $(\frac{3}{5}; \infty)$

$$5x^3(x-1) > 0 \Leftrightarrow \underbrace{5x^2}_{>0} \cdot x \cdot (x-1) > 0 \Leftrightarrow x(x-1) > 0$$

$$(x > 0 \wedge x > 1) \vee (x < 0 \wedge x < 1)$$

$$(1; \infty) \cup (-\infty; 0)$$

$$\Rightarrow \text{ib: } x_0 = 0 \wedge x_0 = 1$$

konvexná na $(-\infty; 0)$ a na $(1; \infty)$

konkávna na $(0; 1)$

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g) $f: y = x^4 + 2x^3 + 6x^2$

$$y'' = (4x^3 + 6x^2 + 12x)' = 12x^2 + 12x + 12 = 0$$

$$x^2 + x + 1 = 0$$

$D = 1 - 4 = -3 \Rightarrow \nexists$ inflexný bod \nearrow

postupujeme tak, že dosadíme do vzťahu ľubovoľné číslo a zistíme, aké znamienko má výsledok; platí: ak $D < 0 \Rightarrow$ funkcia je vždy > 0 alebo < 0 , teda aké znamienko bude mať výsledok, také znamienko má funkcia po dosadení všetkých $x \in D(f)$:

$$x^2 + x + 1 \rightarrow \text{dosadíme } 0 = 0^2 + 0 + 1 = 1 > 0$$

$\Rightarrow y'' > 0 \forall x \in D(f) \Rightarrow$ ib \nexists
funkcia je konvexná na $(-\infty; \infty)$

h) $f: y = x + \frac{1}{x}$; $D(f) = (-\infty; 0) \cup (0; \infty)$

$$y'' = (1 - x^{-2})' = 2x^{-3} = \frac{2}{x^3} = 0 \rightarrow \text{nevrastane nikdy}$$

$\Rightarrow \nexists$ ib

$$\frac{2}{x^3} > 0 \Leftrightarrow x^3 > 0 \Leftrightarrow \underline{x > 0}$$

\Rightarrow ib \nexists
 konvexná na $(0; \infty)$
 konkávna na $(-\infty; 0)$

i) $f: y = 3x + \frac{1}{2x^2}$; $D(f) = (-\infty; 0) \cup (0; \infty)$

$$y'' = (3x + \frac{1}{2} \cdot x^{-2})'' = (3 + \frac{1}{2} \cdot (-2)x^{-3})' = (3 - x^{-3})' = 3x^{-4} = \frac{3}{x^4} \neq 0 \text{ nikdy ; } \frac{3}{x^4} > 0 \text{ vždy} \Rightarrow$$

ib \nexists
konvexná na celom $D(f)$

j) $f: y = \frac{3x^2}{1-x}$; $D(f) = (-\infty; 1) \cup (1; \infty)$

$$y' = \frac{6x \cdot (1-x) - 3x^2 \cdot (-1)}{(1-x)^2} = \frac{6x - 6x^2 + 3x^2}{(1-x)^2} = \frac{6x - 3x^2}{(1-x)^2}$$

$$y'' = \frac{(6-6x) \cdot (1-x)^2 - (6x-3x^2) \cdot 2 \cdot (1-x) \cdot (-1)}{(1-x)^4} = \frac{6 \cdot (1-x)^3 - 3x(2-x) \cdot (-2) \cdot (1-x)}{(1-x)^4} = \frac{(1-x) \cdot [6(1-x)^2 + 6(2-x)]}{(1-x)^4} = \otimes$$

$$\otimes = \frac{6 \cdot (1 - 2x + x^2 + 2 - x)}{(1-x)^3} = \frac{6 \cdot (x^2 - 3x + 3)}{(1-x)^3} \neq 0 \text{ nikdy, lebo } x^2 - 3x + 3 \neq 0, \text{ lebo } D = 9 - 4 \cdot 3 < 0$$

dosadíme $Q: 0^2 - 3 \cdot 0 + 3 > 0 \quad \forall x \in D(f)$

$$\frac{6(x^2 - 3x + 3)}{(1-x)^3} > 0 \Leftrightarrow \underbrace{\frac{6(x^2 - 3x + 3)}{(1-x)^2}}_{> 0} \cdot \frac{1}{1-x} > 0 \Leftrightarrow \frac{1}{1-x} > 0$$

$$\Leftrightarrow 1-x > 0 \Leftrightarrow \underline{x < 1}$$

\Rightarrow ib \nexists
 konvexná na $(-\infty; 1)$
 konkávna na $(1; \infty)$

6.1 k) $f: y = \frac{x^2 + x + 21}{x+2}; \quad D(f) = (-\infty; -2) \cup (-2; \infty)$

$$y' = \frac{(2x+1)(x+2) - (x^2+x+21) \cdot 1}{(x+2)^2} = \frac{2x^2+4x+x+2-x^2-x-21}{(x+2)^2} = \frac{x^2+4x-19}{(x+2)^2}$$

$$y'' = \frac{(2x+4) \cdot (x+2)^2 - (x^2+4x-19) \cdot 2(x+2)}{(x+2)^4} = \frac{2 \cdot (x+2)^3 - 2(x^2+4x-19)(x+2)}{(x+2)^4} = \frac{2(x+2) \cdot [(x+2)^2 - (x^2+4x-19)]}{(x+2)^4}$$

$$= \frac{2 \cdot (x^2+4x+4 - x^2-4x+19)}{(x+2)^3} = \frac{46}{(x+2)^3} \neq 0 \text{ nikdy; } \frac{46}{(x+2)^3} > 0 \Leftrightarrow \underbrace{\frac{46}{(x+2)^2}}_{> 0} \cdot \frac{1}{x+2} > 0 \Leftrightarrow x+2 > 0$$

$$\underline{x > -2}$$

\Rightarrow ib \nexists
 konvexná na $(-2; \infty)$
 konkávna na $(-\infty; -2)$

6.1 b) $f: y = \frac{2x}{1+x^2}$; $D(f) = \mathbb{R}$

$$y' = \frac{2 \cdot (1+x^2) - 2x \cdot 2x}{(1+x^2)^2} = \frac{2+2x^2-4x^2}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$$

$$y'' = \frac{(-4x)(1+x^2)^2 - (2-2x^2) \cdot 2 \cdot (1+x^2) \cdot 2x}{(1+x^2)^4} = \frac{(1+x^2) \cdot [-4x(1+x^2) - 4x(2-2x^2)]}{(1+x^2)^4} = \frac{-4x-4x^3-8x+8x^3}{(1+x^2)^3} = \frac{4x^3-12x}{(1+x^2)^3} =$$

$$= \frac{4x(x^2-3)}{(1+x^2)^3} = 0 \Leftrightarrow \boxed{x_0=0} \wedge x^2-3=0$$

$$\boxed{x_0=\pm\sqrt{3}}$$

$$\frac{4x(x^2-3)}{(1+x^2)^3} > 0 \Leftrightarrow \underbrace{\frac{4}{(1+x^2)^3}}_{>0} \cdot x(x^2-3) > 0 \Leftrightarrow$$

$$\Leftrightarrow x(x-\sqrt{3})(x+\sqrt{3}) > 0$$

	$(-\infty; -\sqrt{3})$	$(-\sqrt{3}; 0)$	$(0; \sqrt{3})$	$(\sqrt{3}; \infty)$
x	-	-	+	+
$x-\sqrt{3}$	-	-	-	+
$x+\sqrt{3}$	-	+	+	+
	-	⊕	-	⊕

$$\Rightarrow \text{ib s\u00fa } x_0=0 \wedge x_0=-\sqrt{3} \wedge x_0=\sqrt{3}$$

konvexn\u00e1 na $(-\sqrt{3}; 0)$ a na $(\sqrt{3}; \infty)$

konk\u00e1vna na $(-\infty; -\sqrt{3})$ a na $(0; \sqrt{3})$



6.1 m) $f: y = \frac{x}{1-x^2}$; $D(f) = (-\infty; -1) \cup (-1; 1) \cup (1; \infty) = \mathbb{R} \setminus \{-1; 1\}$

$$y' = \frac{1 \cdot (1-x^2) - x \cdot (-2x)}{(1-x^2)^2} = \frac{1-x^2+2x^2}{(1-x^2)^2} = \frac{x^2+1}{(1-x^2)^2}$$

$$y'' = \frac{2x \cdot (1-x^2)^2 - (x^2+1) \cdot 2 \cdot (1-x^2) \cdot (-2x)}{(1-x^2)^4} = \frac{(1-x^2) \cdot [2x(1-x^2) + 4x(x^2+1)]}{(1-x^2)^4} = \frac{2x - 2x^3 + 4x^3 + 4x}{(1-x^2)^3} =$$

$$= \frac{2x^3 + 6x}{(1-x^2)^3} = \frac{2x(x^2+3)}{(1-x^2)^3} = \underbrace{\frac{2(x^2+3)}{(1-x^2)^2}}_{>0} \cdot \frac{x}{1-x^2} = 0 \Leftrightarrow \underline{x_0 = 0}$$

$$\frac{2(x^2+3)}{(1-x^2)^2} \cdot \frac{x}{(1-x^2)} > 0 \Leftrightarrow \frac{x}{(1-x^2)} > 0 \Leftrightarrow \frac{x}{(1-x)(1+x)} > 0$$

	$(-\infty; -1)$	$(-1; 0)$	$(0; 1)$	$(1; \infty)$
x	-	-	+	+
$1-x$	+	+	+	-
$1+x$	-	+	+	+
	⊕	-	⊕	-

⇒ ib je $x_0 = 0$

Konveksná na $(-\infty; -1)$ a na $(0; 1)$

Konkávna na $(-1; 0)$ a na $(1; \infty)$



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$$m) f = y = \frac{x^2}{16-x^2}; \quad D(f) = \mathbb{R} \setminus \{-4, 4\}$$

$$y' = \frac{2x(16-x^2) - x^2(-2x)}{(16-x^2)^2} = \frac{32x - 2x^3 + 2x^3}{(16-x^2)^2} = \frac{32x}{(16-x^2)^2}$$

$$y'' = \frac{32(16-x^2)^2 - 32x(16-x^2) \cdot 2 \cdot (-2x)}{(16-x^2)^4} = \frac{(16-x^2) \cdot [32 \cdot (16-x^2) + 32 \cdot 4x^2]}{(16-x^2)^4} = \frac{32 \cdot (16-x^2 + 4x^2)}{(16-x^2)^3} = \frac{32 \cdot (3x^2 + 16)}{(16-x^2)^3}$$

$$= \frac{32 \cdot (3x^2 + 16)}{(16-x^2)^3} = 0 \Leftrightarrow 3x^2 + 16 = 0, \text{ ale } 3x^2 + 16 > 0 \text{ vždy}$$

$$\frac{32(3x^2+16)}{(16-x^2)^3} > 0 \Leftrightarrow \underbrace{\frac{32(3x^2+16)}{(16-x^2)^2}}_{>0} \cdot \frac{1}{16-x^2} > 0 \Leftrightarrow \frac{1}{16-x^2} > 0 \Leftrightarrow (4-x)(4+x) > 0$$

$$(4-x > 0 \wedge 4+x > 0) \vee (4-x < 0 \wedge 4+x < 0)$$

$$(x < 4 \wedge x > -4) \vee (x > 4 \wedge x < -4)$$

$$(-4; 4)$$

$$\emptyset$$

$$\Rightarrow \text{ib } \nexists$$

konvexná na $(-4; 4)$

konkávna na $(-\infty; -4)$ a na $(4; \infty)$

6.2 a) $f: y = \frac{x^2+1}{x^2-1}$; $D(f) = \mathbb{R} \setminus \{-1; 1\}$

$$y' = \frac{2x(x^2-1) - (x^2+1) \cdot 2x}{(x^2-1)^2} = \frac{2x \cdot [x^2-1-x^2-1]}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

$$y'' = \frac{-4 \cdot (x^2-1)^2 - (-4x) \cdot 2(x^2-1) \cdot 2x}{(x^2-1)^4} = \frac{(x^2-1) \cdot [-4(x^2-1) + 16x^2]}{(x^2-1)^4} = \frac{-4x^2+4+16x^2}{(x^2-1)^3} = \frac{12x^2+4}{(x^2-1)^3} = 0 \Leftrightarrow 12x^2+4=0, \text{ ale } 12x^2+4 > 0 \text{ vždy}$$

$$\frac{12x^2+4}{(x^2-1)^3} > 0 \Leftrightarrow \underbrace{\frac{12x^2+4}{(x^2-1)^2}}_{>0} \cdot \frac{1}{x^2-1} > 0 \Leftrightarrow \frac{1}{x^2-1} > 0 \Leftrightarrow x^2-1 > 0 \Leftrightarrow (x-1)(x+1) > 0$$

$$(x > 1 \wedge x > -1) \vee (x < 1 \wedge x < -1)$$

$$(1; \infty) \cup (-\infty; -1)$$

\Rightarrow ib \nexists
 konvexná na $(-\infty; -1)$ a na $(1; \infty)$
 konkávná na $(-1; 1)$

b) $f: y = \frac{1}{x^3} + \frac{1}{x^2}$; $D(f) = \mathbb{R} \setminus \{0\}$

$$y' = \left(x^{-3} + x^{-2}\right)' = \left(-3x^{-4} - 2x^{-3}\right)' = 12x^{-5} + 6x^{-4} = \frac{12}{x^5} + \frac{6}{x^4} = \frac{12+6x}{x^5} = 0 \Leftrightarrow 12+6x=0 \Leftrightarrow \underline{x_0 = -2}$$

$$\frac{12+6x}{x^5} > 0 \Leftrightarrow \underbrace{\frac{1}{x^4}}_{>0} \cdot \frac{12+6x}{x} > 0 \Leftrightarrow \frac{12+6x}{x} > 0 \Leftrightarrow (12+6x > 0 \wedge x > 0) \vee (12+6x < 0 \wedge x < 0)$$

$$(x > -2 \wedge x > 0) \vee (x < -2 \wedge x < 0)$$

$$(0; \infty) \cup (-\infty; -2)$$

\Rightarrow ib je $x_0 = -2$
 konvexná na $(-\infty; -2)$ a na $(0; \infty)$; konkávná na $(-2; 0)$

c) $f: y = \frac{1}{x^3} - \frac{6}{x}$; $D(f) = \mathbb{R} \setminus \{0\}$

$y'' = (x^{-3} - 6x^{-1})'' = (-3x^{-4} + 6x^{-2})' = 12x^{-5} - 12x^{-3} = \frac{12}{x^5} - \frac{12}{x^3} = \frac{12 - 12x^2}{x^5} = 0 \Leftrightarrow 12 - 12x^2 = 0 \Leftrightarrow x^2 = 1$
 $\boxed{x_0 = -1 \wedge x_0 = 1}$

$\frac{12 - 12x^2}{x^5} > 0 \Leftrightarrow \underbrace{\frac{12}{x^4}}_{>0} \cdot \frac{1 - x^2}{x} > 0 \Leftrightarrow \frac{(1-x)(1+x)}{x} > 0$

	$(-\infty; -1)$	$(-1; 0)$	$(0; 1)$	$(1; \infty)$
x	-	-	+	+
$1-x$	+	+	+	-
$1+x$	-	+	+	+
	\oplus	-	\oplus	-

\Rightarrow ib je $x_0 = 1 \wedge x_0 = -1$
 konvexná na $(-\infty; -1)$ a na $(0; 1)$
 konkávna na $(-1; 0)$ a na $(1; \infty)$

d) $f: y = \left(\frac{1}{2} + \frac{1}{x}\right)^2$; $D(f) = \mathbb{R} \setminus \{0\}$

$y' = \left[\left(\frac{x+2}{2x}\right)^2\right]' = 2 \cdot \frac{x+2}{2x} \cdot \frac{1 \cdot 2x - (x+2) \cdot 2}{(2x)^2} = \frac{2(x+2)(2x-2x-4)}{(2x)^3} = \frac{-8(x+2)}{(2x)^3} = \frac{-8x-16}{(2x)^3}$

$y'' = \frac{-8(2x)^3 - (-8x-16) \cdot 3 \cdot (2x)^2 \cdot 2}{[(2x)^3]^2} = \frac{(2x)^2 \cdot [-8 \cdot 2x + 6 \cdot (8x+16)]}{(2x)^6} = \frac{-16x + 48x + 96}{(2x)^4} = \frac{32x + 96}{(2x)^4} = 0 \Leftrightarrow 32x = -96$
 $\boxed{x_0 = -3}$

$\frac{32x + 96}{(2x)^4} = \frac{32}{(2x)^4} \cdot \frac{x+3}{1} > 0 \Leftrightarrow x+3 > 0 \Leftrightarrow \boxed{x > -3} \Rightarrow$

ib je $x_0 = -3$
 konvexná na $(-3; 0)$ a na $(0; \infty)$
 konkávna na $(-\infty; -3)$

(6.2)

e) $f: y = 3x - \sqrt{x-3}$; $D(f): x-3 \geq 0$
 $x \geq 3 \Rightarrow D(f) = \boxed{[3; \infty)}$

$$y'' = \left(3 - \frac{1}{2}(x-3)^{-\frac{1}{2}}\right)' = \frac{1}{4}(x-3)^{-\frac{3}{2}} = \frac{1}{4 \cdot \sqrt{(x-3)^3}} > 0 \text{ v\u0161dy} \Rightarrow \text{\textcancel{ib} konvexn\u00e1 na } \underline{\underline{[3; \infty)}}$$

$$a^x \cdot b^x = (a \cdot b)^x$$

$$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$$

$$(a^x)^y = a^{x \cdot y}$$

$$a^x \cdot a^y = a^{x+y}$$

$$\frac{a^x}{a^y} = a^{x-y}$$

f) $f: y = 4 + \sqrt[3]{x^2}$; ~~$D(f) = \mathbb{R}$~~ $D(f) = \mathbb{R}$

$$y' = \left(\frac{2}{3}x^{-\frac{1}{3}}\right)' = -\frac{2}{9}x^{-\frac{4}{3}} = \frac{-2}{9 \cdot \sqrt[3]{x^4}} \rightsquigarrow \left. \begin{array}{l} \text{\u017etatel } < 0 \text{ v\u0161dy} \\ \text{menovatel } > 0 \text{ v\u0161dy} \end{array} \right\} \frac{-2}{9 \cdot \sqrt[3]{x^4}} < 0 \text{ v\u0161dy} \Rightarrow \text{\textcancel{ib} konk\u00e1vna na } \underline{\underline{(-\infty; \infty)}}$$

g) $f: y = \frac{2x}{\sqrt{x^2+1}}$; $D(f) = \mathbb{R}$

$$y'' = \frac{2 \cdot (x^2+1)^{\frac{1}{2}} - 2x \cdot \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x}{(\sqrt{x^2+1})^2} = \frac{2 \cdot \sqrt{x^2+1} - 2x^2 \cdot \frac{1}{\sqrt{x^2+1}}}{x^2+1} = \frac{2 \cdot (x^2+1) - 2x^2}{\sqrt{x^2+1} \cdot (x^2+1)} = \frac{2x^2+2-2x^2}{(x^2+1) \cdot \sqrt{x^2+1}} = \frac{2}{(x^2+1) \cdot (x^2+1)^{\frac{1}{2}}} = \frac{2}{(x^2+1)^{\frac{3}{2}}}$$

$$y'' = \frac{-2 \cdot \frac{3}{2} \cdot (x^2+1)^{\frac{1}{2}} \cdot 2x}{[(x^2+1)^{\frac{3}{2}}]^2} = \frac{-6x \cdot (x^2+1)^{\frac{1}{2}}}{(x^2+1)^3} = \frac{-6x}{(x^2+1)^{\frac{5}{2}}} = \frac{-6}{(x^2+1)^2} \cdot \frac{x}{(x^2+1)^{\frac{1}{2}}} = \frac{-6}{(x^2+1)^2 \cdot \sqrt{x^2+1}} \cdot x = 0 \Leftrightarrow \underline{x=0}$$

$$\frac{-6 < 0}{(x^2+1)^2 \cdot \sqrt{x^2+1}} \cdot x > 0 \Leftrightarrow x < 0, \text{ lebo [z\u00e1porn\u00e9 \u0165.]} \cdot x > 0 \Leftrightarrow x < 0 \Rightarrow$$

ib je $x_0=0$
 na $(-\infty; 0)$ je konvexn\u00e1
 na $(0; \infty)$ je konk\u00e1vna

(6.3)

(6.2)

$$h) f: y = \frac{x}{\sqrt{x^3+1}}; D(f): \underbrace{x^3+1 \geq 0 \wedge \sqrt{x^3+1} \neq 0}_{x^3+1 > 0}$$

$$x^3 > -1 \Leftrightarrow x > -1 \Rightarrow D(f) = (-1; \infty)$$

$$y' = \frac{1 \cdot \frac{1}{2}(x^3+1)^{-\frac{1}{2}} - x \cdot \frac{1}{2}(x^3+1)^{-\frac{3}{2}} \cdot 3x^2}{x^3+1} = \frac{\frac{1}{2}(x^3+1)^{\frac{1}{2}} - \frac{3}{2}x^3(x^3+1)^{-\frac{1}{2}}}{x^3+1} = \frac{\frac{1}{2} \cdot (x^3+1) - \frac{3}{2}x^3}{\sqrt{x^3+1}} = \frac{-x^3 + \frac{1}{2}}{(x^3+1)^{\frac{3}{2}}}$$

$$y'' = \frac{(-3x^2)(x^3+1)^{\frac{3}{2}} - (-x + \frac{1}{2}) \cdot \frac{3}{2} \cdot (x^3+1)^{\frac{1}{2}} \cdot 3x^2}{[(x^3+1)^{\frac{3}{2}}]^2} = \frac{(x^3+1)^{\frac{1}{2}} \cdot [-3x^2 \cdot (x^3+1) - \frac{9}{2}x^2 \cdot (-x + \frac{1}{2})]}{(x^3+1)^3} = \frac{-3x^5 - 3x^2 + \frac{9}{2}x^5 - \frac{9}{4}x^2}{(x^3+1)^{\frac{5}{2}}} =$$

$$= \frac{x^5 \cdot (-\frac{6}{2} + \frac{9}{2}) + x^2 \cdot (-\frac{12}{4} - \frac{9}{4})}{(x^3+1)^{\frac{5}{2}}} = \frac{\frac{3}{2}x^5 - \frac{21}{4}x^2}{(x^3+1)^{\frac{5}{2}}} = \frac{\frac{3}{4}x^2 \cdot (2x^3 - 7)}{(x^3+1)^2 \cdot \sqrt{x^3+1}} = \frac{\overset{10}{\frac{3}{4}}x^2}{\underset{0}{(x^3+1)^2} \cdot \underset{0}{\sqrt{x^3+1}}} \cdot (2x^3 - 7) = 0 \Leftrightarrow$$

$$\Leftrightarrow x^2 = 0 \quad \vee \quad 2x^3 - 7 = 0$$

$$\boxed{x_0 = 0} \quad \vee \quad x^3 = \frac{7}{2}$$

$$\boxed{x_0 = \sqrt[3]{\frac{7}{2}}}$$

$$y'' > 0 \Leftrightarrow 2x^3 - 7 > 0$$

$$x^3 > \frac{7}{2}$$

$$\boxed{x > \sqrt[3]{\frac{7}{2}}}$$

\Rightarrow ib sú $x_0 = 0 \wedge x_0 = \sqrt[3]{\frac{7}{2}}$
 konvexná na $(\sqrt[3]{\frac{7}{2}}; \infty)$
 konkávna na $(-1; 0) \cup (0; \sqrt[3]{\frac{7}{2}})$

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a) f: y = x · e^{-x}

y'' = (x · e^{-x})'' = [1 · e^{-x} + x · e^{-x} · (-1)]' = [e^{-x} · (1-x)]' = e^{-x} · (-1) · (1-x) + e^{-x} · (-1) = e^{-x} · (-1 · (1-x) - 1) = e^{-x} · (-1 + x - 1) = e^{-x} · (x-2) = 0 ⇔

⇔ x₀ = 2

y'' > 0 ⇔ $\frac{x-2}{e^{-x}} > 0$ ⇔ x-2 > 0 ⇔ x > 2 ⇒
ib je x₀ = 2
konvexná na (2; ∞)
konkávná na (-∞; 2)

b) f: y = e^{-x²}

y'' = [e^{-x²} · (-2x)]' = e^{-x²} · (-2x) · (-2x) + e^{-x²} · (-2) = e^{-x²} · ((-2x)² - 2) = $\frac{4x^2 - 2}{e^{x^2}} = 0$ ⇔ 4x² - 2 = 0 ⇔ x² = $\frac{1}{2}$
x₀ = ±√ $\frac{1}{2}$

y'' > 0 ⇔ $\frac{4x^2 - 2}{e^{x^2}} > 0$ ⇔ 4x² - 2 > 0
4x² > 2
x² > $\frac{1}{2}$ ⇒ (-∞; -√ $\frac{1}{2}$) ∪ (√ $\frac{1}{2}$; ∞) ⇒
(x - √ $\frac{1}{2}$)(x + √ $\frac{1}{2}$) > 0
...

ib je x₀ = -√ $\frac{1}{2}$ ∧ x₀ = √ $\frac{1}{2}$
konvexná na (-∞; -√ $\frac{1}{2}$) a na (√ $\frac{1}{2}$; ∞)
konkávná na (-√ $\frac{1}{2}$; √ $\frac{1}{2}$)

c) f: y = x · e^{-x²}

y' = 1 · e^{-x²} + x · e^{-x²} · (-2x) = e^{-x²} · (1 - 2x²)

y'' = e^{-x²} · (-2x) · (1 - 2x²) + e^{-x²} · (-4x) = e^{-x²} · (-2x · (1 - 2x²) - 4x) = e^{-x²} · (-2x + 4x³ - 4x) = e^{-x²} · (4x³ - 6x) = $\frac{4x^3 - 6x}{e^{x^2}} = 0$ ⇔

⇔ 4x³ - 6x = 0 ⇔ 2x(2x² - 3) = 0 ⇔ x₀ = 0 ∨
∨ 2x² = 3 ⇔ x₀ = ±√ $\frac{3}{2}$

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$$c) y'' > 0 \Leftrightarrow \frac{4x^3 - 6x}{e^{x^2}} > 0 \Leftrightarrow 4x^3 - 6x > 0 \Leftrightarrow 2x(2x^2 - 3) > 0 \Leftrightarrow 4x(x^2 - \frac{3}{2}) > 0 \Leftrightarrow x \cdot (x - \sqrt{\frac{3}{2}}) \cdot (x + \sqrt{\frac{3}{2}}) > 0$$

ib sú $x_0 = 0$
 $x_0 = -\sqrt{\frac{3}{2}}$
 $x_0 = \sqrt{\frac{3}{2}}$

	$(-\infty; -\sqrt{\frac{3}{2}})$	$(-\sqrt{\frac{3}{2}}; 0)$	$(0; \sqrt{\frac{3}{2}})$	$(\sqrt{\frac{3}{2}}; \infty)$
x	-	-	+	+
$x - \sqrt{\frac{3}{2}}$	-	-	-	+
$x + \sqrt{\frac{3}{2}}$	-	+	+	+
	-	⊕	-	⊕

konvexná na $(-\sqrt{\frac{3}{2}}; 0)$
 a na $(\sqrt{\frac{3}{2}}; \infty)$
 konkávna na $(-\infty; -\sqrt{\frac{3}{2}})$
 a na $(0; \sqrt{\frac{3}{2}})$



d) $f: y = x^2 \cdot e^{-x}$

$$y' = 2x \cdot e^{-x} + x^2 \cdot e^{-x} \cdot (-1) = e^{-x} \cdot (2x - x^2)$$

$$y'' = e^{-x} \cdot (-1) \cdot (2x - x^2) + e^{-x} \cdot (2 - 2x) = e^{-x} \cdot (x^2 - 2x + 2 - 2x) = e^{-x} \cdot (x^2 - 4x + 2) = \frac{x^2 - 4x + 2}{e^x} = 0 \Leftrightarrow x^2 - 4x + 2 = 0$$

$$D = 16 - 4 \cdot 2 = 8$$

$$x_{1,2} = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm \sqrt{4 \cdot 2}}{2} = \frac{4 \pm 2\sqrt{2}}{2}$$

$$= \frac{2 \cdot (2 \pm \sqrt{2})}{2} = 2 \pm \sqrt{2}$$

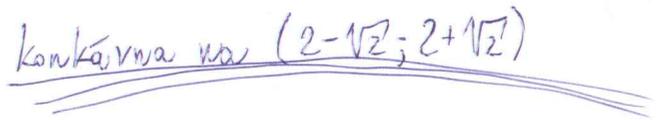
$$\Rightarrow \underline{x_0 = 2 - \sqrt{2} \wedge x_0 = 2 + \sqrt{2}}$$

$$y'' > 0 \Leftrightarrow \frac{x^2 - 4x + 2}{e^x} > 0 \Leftrightarrow (x - 2 + \sqrt{2})(x - 2 - \sqrt{2}) > 0$$

$$(x > 2 + \sqrt{2} \wedge x > 2 - \sqrt{2}) \vee (x < 2 - \sqrt{2} \wedge x < 2 + \sqrt{2})$$

$$(2 + \sqrt{2}; \infty) \cup (-\infty; 2 - \sqrt{2})$$

\Rightarrow ib sú $x_0 = 2 - \sqrt{2}$ a $x_0 = 2 + \sqrt{2}$
 konvexná na $(-\infty; 2 - \sqrt{2})$ a na $(2 + \sqrt{2}; \infty)$
 konkávna na $(2 - \sqrt{2}; 2 + \sqrt{2})$



6.3 e) $f: y = x \cdot e^{\frac{1}{x}}$; $D(f) = \mathbb{R} \setminus \{0\}$

$$y'' = [1 \cdot e^{\frac{1}{x}} + x \cdot e^{\frac{1}{x}} \cdot (-x^{-2})]' = [e^{\frac{1}{x}} \cdot (1 - x^{-1})]' = e^{\frac{1}{x}} \cdot (-x^{-2}) \cdot (1 - x^{-1}) + e^{\frac{1}{x}} \cdot x^{-2} = e^{\frac{1}{x}} \cdot \left(\frac{1 - \frac{1}{x}}{-x^2} + \frac{1}{x^2} \right) = e^{\frac{1}{x}} \cdot \frac{\frac{1}{x} - 1 + 1}{x^2} =$$

$$= e^{\frac{1}{x}} \cdot \frac{1}{x^3} = \frac{e^{\frac{1}{x}}}{x^3} \neq 0 \text{ nikdy}$$

$$y'' > 0 \Leftrightarrow \frac{e^{\frac{1}{x}}}{x^3} > 0 \Leftrightarrow x^3 > 0 \Leftrightarrow x > 0 \Rightarrow \begin{array}{l} \text{konvexná na } (0; \infty) \\ \text{konkávna na } (-\infty; 0) \end{array}$$

f) $f: y = e^{4 - \frac{x^2}{2}}$

$$y'' = [e^{4 - \frac{x^2}{2}} \cdot (-\frac{1}{2} \cdot 2x)]' = (-x \cdot e^{4 - \frac{x^2}{2}})' = -1 \cdot e^{4 - \frac{x^2}{2}} + (-x) \cdot e^{4 - \frac{x^2}{2}} \cdot (-x) = e^{4 - \frac{x^2}{2}} \cdot (-1 + x^2) = (x^2 - 1) \cdot e^{4 - \frac{x^2}{2}} = 0 \Leftrightarrow$$

$$\Leftrightarrow x^2 = 1 \Rightarrow \underline{x_0 = -1 \wedge x_0 = 1}$$

$$y'' > 0 \Leftrightarrow (x^2 - 1) \cdot e^{4 - \frac{x^2}{2}} > 0 \Leftrightarrow x^2 - 1 > 0 \Leftrightarrow x^2 > 1 \Rightarrow (-\infty; -1) \cup (1; \infty)$$

$(x+1)(x-1) > 0$

$$\Rightarrow \text{ib sú } x_0 = -1 \text{ a } x_0 = 1$$

konvexná na $(-\infty; -1)$ a na $(1; \infty)$
konkávna na $(-1; 1)$

g) $f: y = e^{1 - \frac{x^3}{3}}$

$$y'' = [e^{1 - \frac{x^3}{3}} \cdot (-\frac{1}{3} \cdot 3x^2)]' = (e^{1 - \frac{x^3}{3}} \cdot (-x^2))' = e^{1 - \frac{x^3}{3}} \cdot (-x^2) \cdot (-x^2) + e^{1 - \frac{x^3}{3}} \cdot (-2x) = e^{1 - \frac{x^3}{3}} \cdot (x^4 - 2x) = x \cdot e^{1 - \frac{x^3}{3}} \cdot (x^3 - 2) = 0 \Leftrightarrow$$

$$x e^{1 - \frac{x^3}{3}} \cdot (x^3 - 2) > 0 \Leftrightarrow x(x^3 - 2) > 0 \Leftrightarrow (x > 0 \wedge x > \sqrt[3]{2}) \vee (x < 0 \wedge x < \sqrt[3]{2})$$

$$\Leftrightarrow x(x^3 - 2) = 0 \Leftrightarrow \underline{x_0 = 0} \vee \underline{x^3 = 2}$$

$$\underline{x_0 = \sqrt[3]{2}}$$

$(\sqrt[3]{2}; \infty) \cup (-\infty; 0)$

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g) ib sú $x_0 = 0$ a $x_0 = \sqrt[3]{2}$

konvexná na $(-\infty; 0)$ a na $(\sqrt[3]{2}; \infty)$

konkávna na $(0; \sqrt[3]{2})$

h) f: $y = e^{2x} - 8e^x + 5x$

$$y'' = [e^{2x} \cdot 2 - 8e^x + 5]' = 4e^{2x} - 8e^x = \underbrace{4e^x}_{>0} \cdot (e^x - 2) = 0 \Leftrightarrow e^x = 2$$

$$\underline{x_0 = \ln 2}$$

$$4e^x(e^x - 2) > 0 \Leftrightarrow e^x > 2 \Leftrightarrow x > \ln 2 \Rightarrow \text{ib je } x_0 = \ln 2$$

konvexná na $(\ln 2; \infty)$

konkávna na $(-\infty; \ln 2)$

i) f: $y = (2 - x^2) \cdot e^{-x}$

$$y'' = [-2x \cdot e^{-x} + (2 - x^2) \cdot e^{-x} \cdot (-1)]' = [e^{-x} \cdot (-2x - 2 + x^2)]' = [e^{-x} (x^2 - 2x - 2)]' = e^{-x} \cdot (-1) \cdot (x^2 - 2x - 2) + e^{-x} \cdot (2x - 2) =$$

$$= e^{-x} \cdot (-x^2 + 2x + 2 + 2x - 2) = e^{-x} \cdot (-x^2 + 4x) = x e^{-x} (4 - x) = 0 \Leftrightarrow \underline{x_0 = 0} \wedge \underline{x_0 = 4}$$

$$y'' > 0 \Leftrightarrow x(4-x)e^{-x} > 0 \Leftrightarrow (x > 0 \wedge x < 4) \vee (x < 0 \wedge x > 4) \Rightarrow$$

$$(0; 4)$$

ib sú $x_0 = 0$ a $x_0 = 4$

konvexná na $(0; 4)$

konkávna na $(-\infty; 0)$ a na $(4; \infty)$

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(603) j) $f: y = \frac{e^x}{x}$; $D(f) = \mathbb{R} \setminus \{0\}$

$$y'' = \left(\frac{e^x \cdot x - e^x \cdot 1}{x^2} \right)' = \left(\frac{e^x \cdot (x-1)}{x^2} \right)' = \frac{[e^x \cdot (x-1) + e^x \cdot 1] \cdot x^2 - e^x \cdot (x-1) \cdot 2x}{x^4} = \frac{x^2 \cdot e^x \cdot (x-1+1) - 2x \cdot e^x \cdot (x-1)}{x^4} = \frac{x e^x \cdot (x^2 - 2(x-1))}{x^4} =$$

$$= \frac{e^x \cdot (x^2 - 2x + 2)}{x^3} = 0 \Leftrightarrow x^2 - 2x + 2 = 0$$

$$D = 4 - 4 \cdot 2 < 0 \rightarrow \text{dosadíme } 0: 0^2 - 2 \cdot 0 + 2 = 2 > 0 \Rightarrow x^2 - 2x + 2 > 0 \text{ vždy } (\forall x \in D(f))$$

$$y'' > 0 \Leftrightarrow \frac{e^x (x^2 - 2x + 2)}{x^3} > 0 \Leftrightarrow x^3 > 0 \Leftrightarrow x > 0 \Rightarrow$$

ib \nexists
konvexná na $(0; \infty)$
konkávna na $(-\infty; 0)$

L'Hospitalovo pravidlo:

$$\text{Ak } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{0}{0} \text{ alebo } \frac{\pm \infty}{\pm \infty}, \text{ potom}$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$