

4.1. Napíšte rovnicu dotyčnice a normály ku grafu funkcie f v bode T .

a) $f : y = x^2 - 2x + 2, T = [0, ?]$

e) $f : y = \frac{2x}{x+1}, T = [0, ?]$

b) $f : y = 2x - x^2, T = [1, ?]$

f) $f : y = x + \sqrt{4-x}, T = [3, ?]$

c) $f : y = 1 - \frac{1}{x+1}, T = [0, ?]$

g) $f : y = (x-1) \cdot e^x, T = [1, ?]$

d) $f : y = \sqrt[3]{x+4}, T = [-3, ?]$

h) $f : y = \ln \sin x, T = \left[\frac{\pi}{2}, ? \right]$

4.2. Napíšte rovnicu dotyčnice ku grafu funkcie f , ktorá zviera s osou x uhol 45° .

a) $f : y = 2 \cdot \sqrt{x^2 + 3}$

b) $f : y = \operatorname{arctg} 2x$

4.3. Napíšte rovnicu dotyčnice a normály ku grafu funkcie f , ak dotyčnica t je rovnobežná s danou priamkou p .

a) $f : y = \ln(x+1), p : x - y + 2 = 0$

c) $f : y = x^3 - x, p : 2x - y = 0$

b) $f : y = 3 - 2 \cdot e^{\frac{x}{2}}, p : 2x + 2y - 3 = 0$

d) $f : y = \frac{2x-1}{2-x}, p : 3x - y = 0$

4.4. Dané sú funkcie celkových nákladov a príjmov. Vypočítajte hodnoty marginálnych nákladov, príjmov a zisku pre danú úroveň produkcie x a výsledky ekonomicky interpretujte.

a) $C(x) = 600 + 20x, R(x) = 30x, x = 50$

b) $C(x) = x^2 - 6x + 25, R(x) = 25x - 2x^2, x = 5$

c) $C(x) = 50 + 3x - 0,01x^2, R(x) = 2,5x - 0,005x^2, x = 25$

d) $C(x) = \frac{x^3}{3} - 0,5x^2 + 40, R(x) = 150x - 0,2x^2, x = 10$

e) $C(x) = 100 \cdot e^{0,01x}, R(x) = 10x \cdot e^{0,01x}, x = 100$

4.5. Vypočítajte marginálny dopyt pre jednotkovú cenu p , ak je daná funkcia dopytu d . Výsledky ekonomicky interpretujte.

a) $d : q = 56 - 2p, p = 6$

c) $d : q = 1 + 3e^{-\frac{p}{3}}, p = 3$

b) $d : q = 100 - p^2, p = 4$

d) $d : q = \frac{60}{p+3} - 2, p = 2$

4.6. Daná je dopytová funkcia d . Vypočítajte elasticitu dopytu pre jednotkovú cenu p . Výsledky ekonomicky interpretujte.

a) $d : q = 27 - 3p, p = 3$

c) $d : q = \frac{12}{p} - 3, p = 2$

b) $d : q = \frac{32 - p^2}{2}, p = 4$

d) $d : q = 80 - 30\sqrt{p}, p = 4$

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a) $f: y = x^2 - 2x + 2 ; T \in [0; ?]$

~ najprv zistíme y-ovú súradnicu bodu T:

$$y = 0^2 - 2 \cdot 0 + 2 = 2 \quad (\text{dosadili sme } x\text{-ovú súradnicu do predpisu funkcie})$$

$T \in [0; 2]$

~ potom zistíme deriváciu funkcie:

$$f' = y' = (x^2 - 2x + 2)' = 2x - 2,$$

~ nакoniec dosadíme do vzorcov:

dotyčnica: $y = 2 + (2 \cdot 0 - 2) \cdot (x - 0) = 2 + (-2) \cdot x = 2 - 2x = -2x + 2$

$$\underline{\underline{y = -2x + 2}}$$

normála: $y = 2 + \frac{-1}{2 \cdot 0 - 2} \cdot (x - 0) = 2 + \frac{-1}{-2} \cdot x = 2 + \frac{1}{2}x$

$$\underline{\underline{y = \frac{1}{2}x + 2}}$$

b) $f: y = 2x - x^2 ; T = [1; ?]$

$$y = 2 \cdot 1 - 1^2 = 1 \Rightarrow \underline{\underline{T[1; 1]}}$$

$$f' = y' = 2 - 2x$$

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Vzorce:

~ rovnica dotyčnice: $y = f(a) + f'(a) \cdot (x-a)$
 v bode $A[a, f(a)]$

~ rovnica normály: $y = f(a) + \frac{-1}{f'(a)} \cdot (x-a)$
 v bode $A[a, f(a)]$

\nearrow ovač súr. T \nearrow v x-ovej súr. T

dotyčnica: $y = 1 + (2 - 2 \cdot 1) \cdot (x - 1) = 1 + 0 \cdot (x - 1) = 1$

$$\underline{\underline{y = 1}}$$

normála: $y = 1 + \frac{-1}{2 - 2 \cdot 1} \cdot (x - 1) = 1 - \frac{1}{0} \cdot (x - 1) \rightsquigarrow \frac{1}{0} \text{ nemá zmysel}$

$$\underline{\underline{\text{normála } \exists}}$$

dotyčnica: $y = 1 + (2 - 2 \cdot 1) \cdot (x - 1) = 1 + 0 \cdot (x - 1) = 1$

$$\underline{\underline{y = 1}}$$

normála: $y = 1 + \frac{-1}{2 - 2 \cdot 1} \cdot (x - 1) = 1 - \frac{1}{0} \cdot (x - 1) \rightsquigarrow \frac{1}{0} \text{ nemá zmysel}$

$$\underline{\underline{\text{normála } \exists}}$$

4.1

c) $f: y = 1 - \frac{1}{x+1} ; T = [0; ?]$

$$y(0) = 1 - \frac{1}{0+1} = 0 \Rightarrow T[0; 0]$$

$$y' = -\frac{-(x+1)^2}{(x+1)^2} = \underbrace{\frac{+1}{(x+1)^2}}$$

dotyčnica:

$$y = 0 + \frac{1}{(0+1)^2} \cdot (x-0) = \frac{1}{1} \cdot x$$

$$\underline{\underline{y=x}}$$

normála:

$$y = 0 + \frac{-1}{\frac{1}{(0+1)^2}} \cdot (x-0) = -\frac{1}{1} \cdot (x-0) = -x$$

$$\underline{\underline{y=-x}}$$

d) $f: y = \sqrt[3]{x+4} ; T = [-3; ?]$

$$y(-3) = \sqrt[3]{-3+4} = 1 \Rightarrow T[-3; 1]$$

$$y' = [(x+4)^{\frac{1}{3}}]' = \frac{1}{3}(x+4)^{-\frac{2}{3}} \cdot (x+4)' = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{(x+4)^2}} \cdot 1 = \underbrace{\frac{1}{3 \cdot \sqrt[3]{(x+4)^2}}}$$

dotyčnica:

$$y = 1 + \frac{1}{3 \cdot \sqrt[3]{(-3+4)^2}} \cdot (x+3) = 1 + \frac{1}{3} \cdot (x+3) \\ = 1 + \frac{1}{3}x + 1$$

$$\underline{\underline{y = \frac{1}{3}x + 2}}$$

e) $f: y = \frac{2x}{x+1} ; T[0; ?]$

$$y(0) = \frac{2 \cdot 0}{0+1} = 0 \Rightarrow T[0; 0]$$

$$f' = \frac{2 \cdot (x+1) - 2x \cdot 1}{(x+1)^2} = \frac{2x+2-2x}{(x+1)^2} = \underbrace{\frac{2}{(x+1)^2}}$$

$$\text{normála: } y = 1 + \frac{-1}{\frac{2}{(0+1)^2}} \cdot (x+3) = 1 - 3 \cdot (x+3) \\ = 1 - 3x - 9$$

$$\underline{\underline{y = -3x - 8}}$$

4.1 e) dotyčnica: $y = 0 + \frac{2}{(0+1)^2} \cdot (x-0) = \frac{2}{1} \cdot x \Rightarrow \underline{\underline{y=2x}}$

normála: $y = 0 + \frac{-1}{\frac{2}{1}} \cdot (x-0) = -\frac{1}{2}x \Rightarrow \underline{\underline{y=-\frac{1}{2}x}}$

f) $f: y = x + \sqrt{4-x}$; $T = [3; ?]$

$$y(3) = 3 + \sqrt{4-3} = 4 \Rightarrow T[3; 4]$$

$$y' = 1 + \left[(4-x)^{\frac{1}{2}} \right]' = 1 + \frac{1}{2}(4-x)^{-\frac{1}{2}} \cdot (-1) = 1 + \frac{1}{2} \cdot \frac{1}{\sqrt{4-x}} \cdot (-1) = 1 - \underbrace{\frac{1}{2\sqrt{4-x}}}$$

$$\begin{aligned} \text{dotyčnica: } y &= 4 + 1 - \frac{1}{2\sqrt{4-3}} \cdot (x-3) = \\ &= 5 - \frac{1}{2} \cdot (x-3) = 5 - \frac{1}{2}x + \frac{3}{2} \end{aligned}$$

$$\underline{\underline{y = -\frac{1}{2}x + \frac{13}{2}}}$$

g) $f: y = (x-1) \cdot e^x$; $T = [1; ?]$

$$y(1) = 0 \cdot e = 0 \Rightarrow T[1; 0]$$

$$f = 1 \cdot e^x + (x-1) \cdot e^x = e^x \cdot (1+x-1) = \underline{\underline{x \cdot e^x}}$$

dotyčnica: $y = 0 + 1 \cdot e^1 \cdot (x-1) = e \cdot (x-1) = ex - e$

$$\underline{\underline{y = ex - e}}$$

normála: $y = 0 + \frac{-1}{e} \cdot (x-1) = -\frac{1}{e}x + \frac{1}{e}$

$$\underline{\underline{y = -\frac{1}{e}x + \frac{1}{e}}}$$

$$\underline{\underline{y = -2x + 10}}$$

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(4.1)

$$h) f: y = \ln(\sin x); T = \left[\frac{\pi}{2}; ? \right]$$

$$y\left(\frac{\pi}{2}\right) = \ln(\sin \frac{\pi}{2}) = \ln 1 = 0 \Rightarrow T\left[\frac{\pi}{2}; 0\right]$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x} = \underline{\operatorname{cotg} x}$$

dotyčnica:

$$y = 0 + \operatorname{cotg} \frac{\pi}{2} \cdot \left(x - \frac{\pi}{2}\right) = 0 \cdot \left(x - \frac{\pi}{2}\right) = 0$$

$$\underline{\underline{y=0}}$$

normála:

$$y = 0 + \frac{-1}{0} \cdot \left(x - \frac{\pi}{2}\right)$$

$$\underline{\underline{\text{normála } \nexists}}$$

(4.2)

a) $f: y = 2 \cdot \sqrt{x^2+3} ; d = 45^\circ$ platí: $f'(x) = \operatorname{tg} d$

$$y' = 2 \cdot \left[(x^2+3)^{\frac{1}{2}} \right]' = 2 \cdot \frac{1}{2} \cdot (x^2+3)^{-\frac{1}{2}} \cdot 2x = \\ = 2x \cdot \frac{1}{\sqrt{x^2+3}} = \frac{2x}{\sqrt{x^2+3}}$$

$$\frac{2 \cdot x_0}{\sqrt{x_0^2+3}} = \operatorname{tg} 45^\circ = 1 \quad / \cdot \sqrt{x_0^2+3}$$

$$2x_0 = \sqrt{x_0^2+3} \quad |^2$$

$$4x_0^2 = x_0^2 + 3$$

$$3x_0^2 = 3$$

$$x_0^2 = 1$$

$x_0 = \pm 1$ \rightsquigarrow dosadíme do pôvodného predpisu aby sme zistili y_0 :

$$y_0 = 2 \cdot \sqrt{x_0^2+3}$$

$$y_0 = 2 \cdot \sqrt{4} = 4 \Rightarrow T_1 = [-1; 4] \wedge T_2 = [1; 4]$$

dotyčnica:

$$y = 4 + \frac{2 \cdot 1}{\sqrt{1^2+3}} \cdot (x-1) = 4 + \frac{2}{2} \cdot (x-1) = 4+x-1$$

$$\underline{\underline{y=x+3}}$$

normála:

$$y = 4 + \frac{-1}{\frac{2}{2}} \cdot (x-1) = 4-x+1$$

$$\underline{\underline{y=-x+5}}$$

ak ale dosadíme $x_0 = -1$ späť, zistíme že vztah neplatí $\Rightarrow T = [1; 4]$

(4,2)

b) $f: y = \operatorname{arctg} 2x ; \alpha = 45^\circ \Rightarrow \underline{\operatorname{tg} \alpha = 1}$

$$y = (\operatorname{arctg} 2x) = \frac{1}{1+(2x)^2} \cdot 2 = \frac{2}{1+4x^2}$$

$$\frac{2}{1+4x_0^2} = 1 \quad | \cdot (1+4x_0^2)$$

$$2 = 1 + 4x_0^2$$

$$4x_0^2 = 1$$

$$x_0^2 = \frac{1}{4}$$

$$x_0 = \pm \frac{1}{2} \rightarrow \text{když dosadíme späť, zistíme, že obě možnosti vyhovují}$$

zistíme y_0 :

$$y_0 = \operatorname{arctg} 2 \cdot \left(\pm \frac{1}{2}\right) = \operatorname{arctg} (\pm 1)$$

$$\text{ak } x_0 = -\frac{1}{2} \Rightarrow y_0 = \operatorname{arctg} (-1) = -\frac{\pi}{4}$$

$$x_0 = \frac{1}{2} \Rightarrow y_0 = \operatorname{arctg} 1 = \frac{\pi}{4}$$

$$\left. \begin{array}{l} T_1 \left[-\frac{1}{2}; -\frac{\pi}{4} \right] \\ T_2 \left[\frac{1}{2}; \frac{\pi}{4} \right] \end{array} \right\}$$

1. dotyčnice:

$$y = \frac{\pi}{4} + \frac{2}{1+4 \cdot \left(\frac{1}{2}\right)^2} \cdot \left(x - \frac{1}{2}\right) = \frac{\pi}{4} + \frac{2}{2} \cdot \left(x - \frac{1}{2}\right)$$

$$\underline{\underline{y = x - \frac{1}{2} + \frac{\pi}{4}}}$$

$$1. \text{ normála: } y = \frac{\pi}{4} + \frac{-1}{\frac{2}{2}} \cdot \left(x - \frac{1}{2}\right) = \frac{\pi}{4} - x + \frac{1}{2}$$

$$\underline{\underline{y = -x + \frac{1}{2} + \frac{\pi}{4}}}$$

2. dotyčnice:

$$y = -\frac{\pi}{4} + \frac{2}{1+4 \cdot \left(-\frac{1}{2}\right)^2} \cdot \left(x + \frac{1}{2}\right) = -\frac{\pi}{4} + \frac{2}{2} \cdot \left(x + \frac{1}{2}\right)$$

$$\underline{\underline{y = x + \frac{1}{2} - \frac{\pi}{4}}}$$

2. normála:

$$y = -\frac{\pi}{4} + \frac{-1}{\frac{2}{2}} \cdot \left(x + \frac{1}{2}\right) = -\frac{\pi}{4} - x - \frac{1}{2}$$

$$\underline{\underline{y = -x - \frac{1}{2} - \frac{\pi}{4}}}$$

~~(konec)~~

(4.3)

a) $f: y = \ln(x+1)$; $p: y = x+2$; $A \parallel p \rightarrow$ keďže sú dané priamky rovnobežné, majú rovnaké smernice, t.j. číslo pred „ x “ v danom predpise $y = x+2 = 1 \cdot x + 2$ a teda smernica dotyčnice bude tiež 1

$$y = \frac{1}{x+1}$$

$$\frac{1}{x_0+1} = 1$$

$$1 = x_0 + 1$$

$$x_0 = 0$$

$$y_0 = \ln(x_0 + 1) = \ln(0 + 1) = 0 \Rightarrow T[0; 0]$$

dotyčnica:

$$y = 0 + \frac{1}{0+1} \cdot (x - 0) = x \Rightarrow \underline{\underline{y = x}}$$

normála:

$$y = 0 + \frac{-1}{\frac{1}{1}} \cdot (x - 0) \Rightarrow \underline{\underline{y = -x}}$$

b) $f: y = 3 - 2 \cdot e^{\frac{x}{2}}$; $p: 2x + 2y - 3 = 0$; $p \parallel A$

$$2y = -2x + 3$$

$$y = -x + \frac{3}{2}$$

$$y = -2 \cdot e^{\frac{x}{2}} \cdot \frac{1}{2} = -e^{\frac{x}{2}}$$

$$-e^{\frac{x_0}{2}} = -1$$

$$e^{\frac{x_0}{2}} = 1 \quad [e^0 = 1]$$

$$\frac{x_0}{2} = 0$$

$$x_0 = 0 \rightarrow y_0 = 3 - 2 \cdot e^{\frac{0}{2}} = 1 \Rightarrow T[0; 1]$$

dotyčnica:

$$y = 1 + (-e^{\frac{0}{2}}) \cdot (x - 0) = 1 - 1 \cdot x \Rightarrow \underline{\underline{y = -x + 1}}$$

normála:

$$y = 1 + \frac{-1}{-1} \cdot (x - 0) = 1 + x \Rightarrow \underline{\underline{y = x + 1}}$$

(4.3)

c) $f: y = x^3 - x$; $p: y = 2x$; $A \parallel p$

$$\underline{y = 3x^2 - 1}$$

$$3x_0^2 - 1 = 2$$

$$3x_0^2 = 3$$

$$x_0^2 = 1$$

$$x_0 = \pm 1$$

$$\begin{aligned} x_0 = -1 &\Rightarrow y_0 = (-1)^3 - (-1) = -1 + 1 = 0 \\ x_0 = 1 &\Rightarrow y_0 = 1^3 - 1 = 0 \end{aligned} \quad \left. \begin{array}{l} T_1[-1; 0] \\ T_2[+1; 0] \end{array} \right\}$$

1. dotyčnica:

$$y = 0 + (3 \cdot (-1)^2 - 1) \cdot (x + 1) = 2 \cdot (x + 1) \Rightarrow \underline{\underline{y = 2x + 2}}$$

1. normála:

$$y = 0 + \frac{-1}{2} \cdot (x + 1) = -\frac{1}{2} \cdot (x + 1) \Rightarrow \underline{\underline{y = -\frac{1}{2}x - \frac{1}{2}}}$$

2. dotyčnica:

$$y = 0 + (3 \cdot 1^2 - 1) \cdot (x - 1) = 2 \cdot (x - 1) \Rightarrow \underline{\underline{y = 2x - 2}}$$

2. normála:

$$y = 0 + \frac{-1}{2} \cdot (x - 1) = -\frac{1}{2} \cdot (x - 1) \Rightarrow \underline{\underline{y = -\frac{1}{2}x + \frac{1}{2}}}$$

d) $f: y = \frac{2x-1}{2-x}$; $p: y = 3x$; $p \parallel A$

$$y = \frac{2 \cdot (2-x) - (2x-1) \cdot (-1)}{(2-x)^2} = \frac{4-2x+2x-1}{(2-x)^2} = \frac{3}{(2-x)^2}$$

$$\frac{3}{(2-x_0)^2} = 3$$

$$3 = 3 \cdot (2-x_0)^2$$

$$(2-x_0)^2 = 1$$

$$(2-x_0)^2 - 1 = 0 \quad [a^2 - b^2 = (a+b) \cdot (a-b)]$$

$$(2-x_0+1) \cdot (2-x_0-1) = 0 \quad x_{01} = 3$$

$$(3-x_0)(1-x_0) = 0 \Rightarrow x_{02} = 1$$

$$\begin{aligned} x_0 = 3 &\Rightarrow y_0 = \frac{2 \cdot 3 - 1}{2 - 3} = \frac{5}{-1} = -5 \\ x_0 = 1 &\Rightarrow y_0 = \frac{2 \cdot 1 - 1}{2 - 1} = \frac{1}{1} = 1 \end{aligned}$$

$$\Rightarrow \begin{array}{l} T_1[3; -5] \\ T_2[1; 1] \end{array}$$

1. dotyčnica: $y = -5 + \frac{3}{(2-3)^2} \cdot (x-3) = -5 + 3(x-3)$

$$\underline{\underline{y = 3x - 14}}$$

1. normála: $y = -5 + \frac{-1}{3} \cdot (x-3)$

$$\underline{\underline{y = -\frac{1}{3}x - 4}}$$

2. dotyčnica: $y = 1 + \frac{3}{(2-1)^2} \cdot (x-1) = 1 + 3 \cdot (x-1)$

$$\underline{\underline{y = 3x - 2}}$$

2. normála: $y = 1 + \frac{-1}{3} \cdot (x-1)$

$$\underline{\underline{y = -\frac{1}{3}x + \frac{4}{3}}}$$